M.A/M.Sc. I Sem. 2013

Paper — II (Analysis I)

Model/ Suggestive Answers

Paper Code
AS-2218

1 (i) Given $f \in RS(\alpha)$ and $f(x) \geq 0$

Then $\int_a^b f d\alpha \geq 0$ if $b \geq a$

≤ 0 if $b \leq a$

(ii) Let P^* be common refinement of P_1 & P_2 so that

$$P^* = P_1 \cup P_2$$

Now we get

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$

$$\Rightarrow L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Keeping P_2 fixed and taking l.u.b. over all partitions P_1, we get

$$\underline{\int_a^b} f d\alpha \leq U(P_2, f, \alpha)$$

Taking the g.l.b over P_2, we get

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$$

(iii) Algebras of RS- integrable functions:

(a) If $f_1 \in RS(\alpha)$, $f_2 \in RS(\alpha)$ on $[a, b]$, Then

$f_1 + f_2 \in RS(\alpha)$ & $\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$

(b) If $f \in RS(\alpha)$, c is constant; then

$cf \in RS(\alpha)$ and $\int_a^b cf d\alpha = c \int_a^b f d\alpha$

(c) If $f_1 \in RS(\alpha)$, $f_2 \in RS(\alpha)$; $f_1(x) \leq f_2(x)$ on $[a, b]$, Then

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

(d) If f is RS integrable on $[a,b]$, then $|f|$ is also RS- integrable on $[a,b]$

(e) If $f \in RS(\alpha_1)$ and $f \in RS(\alpha_2)$, then

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

and many more theorems may be written by students.

(iv) f is a pointwise limit of a sequence of functions $\{f_n\}$ defined on $[a,b]$, if to each $\epsilon > 0$. and to each $x \in [a,b]$, there corresponds an integer m such that

$$|f_n(x) - f(x)| < \epsilon \qquad \forall\ n \geq m$$

(v) Consider the sequence $\{f_n\}$ as

$$f_n(x) = nx(1-x^2)^n \qquad 0 \leq x \leq 1, \quad n = 1, 2, 3, \ldots$$

For $0 \leq x < 1$ $\displaystyle\lim_{n\to\infty} f_n(x) = 0$

at $x = 0$, each $f_n(0) = 0$ so that $\displaystyle\lim_{n\to\infty} f_n(0) = 0$

Thus $f(x) = \displaystyle\lim_{n\to\infty} f_n(x) = 0$ for $0 \leq x \leq 1$

$$\therefore \int_0^1 f(x)\,dx = 0$$

Again $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$

$$\therefore \lim_{n\to\infty} \left\{ \int_0^1 f_n(x)\,dx \right\} \neq \int_0^1 f\,dx$$

(vi) Given $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}} = \dfrac{\cancel{x^{2n}}\ 1}{\cancel{x^{2n}}\left(1+\frac{1}{x^{2n}}\right)}$

If $|x|>1$, then $\lim\limits_{n\to\infty} \dfrac{1}{x^{2n}} = 0$, Now

$$\lim_{n\to\infty} f_n(x) = \frac{1}{1+0} = 1$$

If $|x|<1$, then $\lim\limits_{n\to\infty} \dfrac{1}{x^{2n}} \Rightarrow \infty$, then

$$\lim_{n\to\infty} f_n(x) = \frac{1}{1+\infty} = 0$$

(vii) Nowhere Convergent Series

If in the power series $\sum\limits_{n=0}^{\infty} a_n x^n$, Then when $\lim |a_n|^{\frac{1}{n}} = \infty$ or $R=0$, the power series is nowhere convergent

Exp. $1+x+\lfloor 2\, x^2 + \lfloor 3\, x^3 + \ldots\ldots$ does not converge for any value of x (other than 0). Its radius of convergence $R=0$.

(viii) The given series
$$\sum_{n=0}^{\infty} \frac{x^n}{\lfloor n} = \sum_{n=0}^{\infty} a_n x^n \quad (\text{Say})$$

then $a_n = \dfrac{1}{\lfloor n}$

Now $R = \lim\limits_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty} \left(\dfrac{\lfloor n+1}{\lfloor n}\right) = \lim\limits_{n\to\infty} (n+1) = \infty$

The power series convergences absolutely for all x

(ix) A power series $\sum_{n=0}^{\infty} a_n x^n$ is continuous function of x within its interval of convergence.

The given series is

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 2.3x + 3.3^2 x^2 + \cdots + n.3^{n-1} x^{n-1} + \cdots$$

Now $a_n = n\,3^{n-1}$

$$\therefore \quad R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n.3^{n-1}}{n+1.\,3^n}\right| = \frac{1}{3}$$

Thus $\left(-\frac{1}{3}, \frac{1}{3}\right)$ lies in circle of $R = \frac{1}{3}$.

Thus $f(x)$ is continuous in $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(x) Szâsz Operators

For $f \in C[0,\infty)$, the Szâsz operator is defined as

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{\lfloor k} f\left(\frac{k}{n}\right)$$

which approximates to $f \in C[0,\infty)$.

(2) Let us suppose that $S(P,f,\alpha)$ exists as $\mu(P) \to 0$ and is equal to A.

Therefore for given $\epsilon > 0, \exists\, \delta > 0$ such that for every partition P of $[a,b]$ with mesh $\mu(P) < \delta$ and for every choice t_i in Δx_i, we have

$$|S(P,f,\alpha) - A| < \frac{\epsilon}{2}$$

or

$$A - \frac{\epsilon}{2} < S(P,f,\alpha) < A + \frac{\epsilon}{2} \quad \text{——— (1)}$$

Let P be one such partition. If we let the points t_i range over the intervals Δx_i and infimum and supremum of sums $S(P,f,\alpha)$, (1) yields

$$A - \frac{\epsilon}{2} < L(P,f,\alpha) \le U(P,f,\alpha) < A + \frac{\epsilon}{2}$$

$$\Rightarrow \quad U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow \quad f \in RS(\alpha) \text{ on } [a,b].$$

Again since $S(P,f,\alpha)$ and $\int_a^b f\,d\alpha$ lie between $U(P,f,\alpha)$ and $L(P,f,\alpha)$

$$\therefore \left| S(P,f,\alpha) - \int_a^b f\,d\alpha \right| \le U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow \lim_{\mu(P)\to 0} S(P,f,\alpha) = \int_a^b f\,d\alpha.$$

Proved

(3) Let $P = \{0, \frac{1}{n}, \frac{2}{n}, \cdots \frac{n}{n} = 1\}$ be partition of $[0,1]$

Also let $I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$ be the r^{th} interval.

Then $M_r = \sup_{x \in I_r} f(x) = \frac{r^2}{n^2}$

$$m_r = \inf_{x \in I_r} f(x) = \frac{(r-1)^2}{n^2}$$

Also $\Delta\alpha_r = \alpha(x_r) - \alpha(x_{r-1})$

$$= \frac{r^2}{n^2} - \frac{(r-1)^2}{n^2} = \frac{2r-1}{n^2}$$

Now $U(P,f,\alpha) = \sum_{r=1}^{n} M_r \Delta\alpha_r = \sum_{r=1}^{n} \frac{r^2}{n^2} \cdot \frac{2r-1}{n^2}$

$$= \frac{1}{n^4}\left\{2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2\right\}$$

$$= \frac{1}{n^4}\left\{2 \cdot \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right\}$$

$$= \frac{3n^3 + 4n^2 + 1}{6n^4}$$

$\therefore \overline{\int_0^1} f d\alpha = \inf \cancel{U}(P,f,\alpha) = \frac{1}{2}$

Now $L(P,f,\alpha) = \sum_{r=1}^{n} m_r \Delta\alpha_r$

$$= \sum_{r=1}^{n} \frac{(r-1)^2}{n^2} \cdot \frac{2r-1}{n^2}$$

$$= \frac{1}{n^4}\left\{ 2\sum_{r=1}^{n}(r-1)^3 + \sum_{r=1}^{n}(r-1)^2 \right\}$$

$$= \frac{3n^3 - 4n^2 + 1}{6n^3}$$

$\therefore \underline{\int_0^1} f d\alpha = \text{Sup } L(P,f,\alpha) = \frac{1}{2}$

$\therefore \underline{\int_0^1} f d\alpha = \overline{\int_0^1} f d\alpha = \frac{1}{2}$

Solved

(4) Dirichlet Test

If $b_n(x)$ is a positive, monotonic decreasing function of n for each value of x in $[a,b]$, if $b_n(x)$ tends uniformly to zero for $a \le x \le b$ and if there is a number k independent of x and n such that for all values of x in $[a,b]$, $\left|\sum_{r=1}^{n} u_r(x)\right| < k \quad \forall n$;

then the series $\sum b_n(x) u_n(x)$ is uniformly convergent for $a \le x \le b$.

Proof Since $b_n(x)$ tends uniformly to zero, therefore for any $\epsilon > 0$, we can find an integer N (independent of x) such that for all values of of x in $[a,b]$

$$0 < b_n(x) < \frac{\epsilon}{2k} \quad \text{for } n \geq N$$

For such values of n and any integral values of $p \geq 1$; we have by Abel's lemma

$$\left| \sum_{r=n+1}^{n+p} b_r(x) u_r(x) \right| \leq b_n(x) \max_{q=1,2,\ldots p} \left| \sum_{r=n+1}^{n+q} u_r(x) \right|$$

$$\leq b_n(x) \left\{ \left| \sum_{r=1}^{n} u_r(x) \right| + \max_{q=1,2,\ldots p} \left| \sum_{r=1}^{n+q} u_r(x) \right| \right.$$

$$< \frac{\epsilon}{2k}(k+k) = \epsilon$$

Hence by Cauchy criterion, the series $\sum b_n(x) u_r(x)$ converges uniformly for for $x \in [a,b]$.

<u>Exp.</u> The series $\sum (-1)^p \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval, but does not converges absolutely for any value of x.

(5) The series $\sum \frac{\cos n\theta}{n^p}$ converges uniformly for all values of $p > 0$ in the interval $[\beta, 2\pi-\beta]$ where $\beta > 0$.

when $p > 1$, then by Weierstrass M Test, the series $\sum \frac{\cos n\theta}{n^p}$ converges uniformly for all real values of θ.

when $0 < p \leq 1$, the series $\sum \frac{\cos n\theta}{n^p}$ uniformly in any interval $[\beta, 2\pi-\beta]$, $\beta > 0$. This can be done by Dirichlet test.

Take $b_n = \frac{1}{n^p}$ and $u_n = \cos n\theta$

$\left(\frac{1}{n^p}\right)$ is positive, monotonic decreasing and tending uniformly to zero $(0<p\le 1)$ and

$$\left|\sum_{r=1}^{n} u_r\right| = \left|\sum_{r=1}^{n} \cos n\theta\right| = \left|\frac{\cos(n+1)\frac{\theta}{2} \cdot \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right|$$

$$< \operatorname{cosec}\frac{\alpha}{2} \quad \forall n$$

Thus all conditions are fullfilled. Hence $\sum \frac{\cos n\theta}{n^p}$ converges uniformly for $p>0$ in interval $[\beta, 2\pi-\beta]$, $\beta>0$

Now taking $p=6$, the given problem is special case of solved one.

(6) Now for $|x|<R$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (x-a+a)^n$$

$$= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} a^{n-m} (x-a)^m$$

We wish to change of order summation

To prove its validity, we notice that it is the summation by rows of a double series which if convergent absolutely, will converge by columns as well as to the same sum.

Replacing all quantities by their modulii and taking all the terms with positive sign, we write

$$\sum_{n=0}^{\infty} |a_n| \sum_{m=0}^{n} \binom{n}{m} |a|^{n-m} |x-a|^m$$

~~$\frac{1}{n!}$ is p~~

$$= \sum_{n=0}^{\infty} |a_n| \left(|x-a| + |a| \right)$$

which is power series and converges for

$|x-a| + |a| < R$

$\Rightarrow |x-a| < R - |a|$

Hence for $|x-a| < R-|a|$, change the order of summation

$$f(x) = \sum_{m=0}^{\infty} \left\{ \sum_{n=m}^{\infty} \binom{n}{m} a^{n-m} \right\} (x-a)^m \qquad (3)$$

Now let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_m x^m + \cdots + a_n x^n + \cdots$$

$$\therefore f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + m a_m x^{m-1} + \cdots + n x^{n-1} + \cdots$$

$$f^{(2)}(x) = \lfloor 2\, a_2 + 3.2\, a_3 x + \cdots m(m-1) a_m x^{m-2} + \cdots + n(n-1) a_n x^{n-2} + \cdots$$

$$f^{(m)}(x) = \lfloor m\, a_m + (m+1)m(m-1) \cdots\cdots 3.2\, a_{m+1} x + \cdots$$

$$+ n(n-1) \cdots\cdots (n-m+1) a_n x^{n-m}$$

$$= \lfloor m \left\{ a_m + \binom{m+1}{m} a_{m+1} x + \binom{m+2}{m} a_{m+2} x^2 + \cdots \right\}$$

$$\therefore f^{(m)}(a) = \lfloor m \sum_{n=m}^{\infty} \binom{n}{m} a_n a^{n-m} \qquad (4)$$

Putting the value of (4) in (3), we get

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{m}(a)}{\lfloor m} (x-a)^m \qquad |x-a| < R-|a|$$

Proved

(7) We know

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \qquad -1<x<1$$

and

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots \qquad -1<x<1$$

Both series are absolutely convergent in $]-1,1[$, therefore can be multiplied and their Cauchy product is

$$(1+x^2)^{-1}\tan^{-1} x = x - \left(1+\frac{1}{3}\right)x^3 + \left(1+\frac{1}{3}+\frac{1}{5}\right)x^5 - \dots \qquad -1<x<1$$

Integrating

$$\frac{1}{2}\left(\tan^{-1} x\right)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1+\frac{1}{3}\right) + \frac{x^6}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right) - \dots \qquad -1<x<1$$

the constant of integration vanishes.

(8) Let f be a function of $C[a,b]$

Given $\epsilon > 0$, we shall obtain the inequality for $|f(t) - (L_n f)(t)|$.

Begin by selecting $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Now put $\alpha = \frac{2\|f\|}{\delta^2}$ and let t be the arbitrary but fixed point of $[a,b]$.

If $|t-x| < \delta \Rightarrow |f(t) - f(x)| < \epsilon$ whereas

$$|t-x| \geq \delta \Rightarrow |f(t) - f(x)| \leq \frac{2\|f\|}{\delta^2}(t-x)^2 = \alpha\,\phi_t(x)$$

Thus for all x, the following is satisfied

$$-\epsilon - \alpha \phi_t(x) \leq f(t) - f(x) \leq \epsilon + \alpha \phi_t(x)$$

In order to write an inequality on functions, let $f_0(x) = 1$. Then we have

$$-\epsilon f_0 - \alpha \phi_t \leq f(t) f_0 - f \leq \epsilon f_0 + \alpha \phi_t$$

By Linearity and monotonicity of L_n,

$$-\epsilon (L_n f_0)(t) - \alpha (L_n \phi_t)(t) \leq f(t)(L_n f_0)(t) - (L_n f)(t)$$
$$\leq \epsilon (L_n f_0)(t) + \alpha (L_n \phi_t)(t)$$

This yields

$$|f(t)(L_n f_0)(t) - (L_n f)(t)| \leq \epsilon \|L_n f_0\| + \alpha (L_n \phi_t)(t) \quad \text{---} \; (3)$$

Now

$$|f(t) - (L_n f)(t)| \leq |f(t) - f(t)(L_n f_0)(t) + f(t)(L_n f_0)(t) - (L_n f)(t)|$$
$$\leq |f(t)|\,|1 - (L_n f_0)(t)| + |f(t)(L_n f_0)(t) - (L_n f)(t)| \quad \boxed{4}$$

Using $\boxed{3}$ in (4), we get

$$|f(t) - (L_n f)(t)| \leq \|f\|\,\|f_0 - L_n f_0\| + \epsilon (1 + \|f_0 - L_n f_0\| + \alpha (L_n \phi_t)(t).$$

This completes the proof.

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